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On Half-Transitive Metacirculant Graphs of Prime-Power Order

Cai Heng Li[1]

Department of Mathematics and Statistics, The University of Western Australia, Nedlands, Western Australia 6097, Australia

and

Hyo-Seob Sim[2]

Division of Mathematical Sciences, Pukyong National University, Pusan 608-737, Korea

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In this paper, infinitely many graphs of large valency which are half-transitive (that is vertex- and edge-transitive but not arc-transitive) are constructed, and a complete classification is given of half-transitive metacirculant graphs of order a p-power and valency less than $2p$, where p is a prime. In particular, it is shown that, for any odd prime p, integers $n \geqslant 3$ and $k \geqslant 2$ such that k divides $p-1$, there are exactly $(p^{n-2}-1)/2+p^{n-3}-1$ nonisomorphic connected half-transitive metacirculants of order p^n and valency $2k$. © 2001 Academic Press

1. INTRODUCTION

Let Γ be a graph with vertex set $V\Gamma$ and edge set $E\Gamma$. Let $\mathrm{Aut}\,\Gamma$ denote the full automorphism group of a graph Γ. If $\mathrm{Aut}\,\Gamma$ is transitive on $V\Gamma$ or $E\Gamma$, then Γ is said to be *vertex-transitive* or *edge-transitive*, respectively. An ordered pair of vertices which are adjacent is called an *arc*. If $\mathrm{Aut}\,\Gamma$ is transitive on the set of arcs of Γ, then Γ is said to be *arc-transitive*. It is clear that an arc-transitive graph is vertex-transitive and edge-transitive. A graph is said to be *half-transitive* if it is vertex- and edge-transitive but not arc-transitive. The investigation of half-transitive graphs was initiated with a question of Tutte [30, p. 60] regarding their existence for even valencies $2k \geqslant 4$. In 1970, Bouwer [9] constructed the first family of half-transitive graphs. Constructing and characterizing half-transitive graphs is currently an active topic in algebraic graph theory (see, for example, [7, 18, 19, 21, 32]).

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For a transitive permutation group G on a set Ω, any orbital graph corresponds to a vertex- and edge-transitive graph Γ with $V\Gamma = \Omega$. If the orbital is non-self-paired, then G is not transitive on the arc set of Γ. However, G may be only a proper subgroup of $\mathrm{Aut}\,\Gamma$, and hence $\mathrm{Aut}\,\Gamma$ may be transitive on the arc set of Γ. Thus, to decide whether Γ is arc-transitive depends not only on the property of G but on that of $\mathrm{Aut}\,\Gamma$. The problem of determining $\mathrm{Aut}\,\Gamma$, however, is in general a very difficult one. So it is unlikely to give a useful characterization of general half-transitive graphs. The present paper will focus on the metacirculant graphs, defined as follows.

DEFINITION 1. An (m, n)-*metacirculant* is a graph of order mn which has an automorphism σ with a cycle decomposition

$$\sigma = (v_{11} v_{12} \cdots v_{1n})(v_{21} v_{22} \cdots v_{2n}) \cdots (v_{m1} v_{m2} \cdots v_{mn})$$

and an automorphism τ normalizing $\langle \sigma \rangle$ and cyclically permuting the orbits

$$V_i = \{v_{1i}, v_{2i}, ..., v_{mi}\}, \qquad i = 1, 2, ..., n,$$

such that τ has a cycle of size m in its disjoint cycle decomposition; also refer to [3].

Metacirculant graphs were introduced by Alspach and Parsons [4] and have many interesting and important properties. This class of graphs has received considerable interest over the years, with a number of research articles covering a wide spectrum of problems including the hamiltonicity problem for vertex-transitive graphs (refer to [1, 2, 5, 26, 27]), the isomorphism problem (refer to [10, 28]), the problem of construction and classification of vertex-transitive graphs of particular orders (refer to [4, 6, 22]), and more recently half-transitivity properties of metacirculants (see [3, 7, 18, 20, 21, 25, 31, 32]). Also, we note that the smallest half-transitive graph is a metacirculant graph of order 27 (refer to [12, 32]). (The authors are grateful to the referee for providing information and references about the literature of the study on metacirculants.)

For a finite group G and a subset S which does not contain the identity 1 such that $S = S^{-1} := \{s^{-1} \mid s \in S\}$, the associated Cayley graph $\mathrm{Cay}(G, S)$ is the graph with vertex set G and edge set $\{\{x, sx\}, \mid x \in G, s \in S\}$. It easily follows that $\mathrm{Cay}(G, S)$ is a graph with valency $|S|$ and that $\mathrm{Cay}(G, S)$ is connected if and only if $\langle S \rangle = G$. Next we construct a class of Cayley graphs.

DEFINITION 2. Let p be a prime and let $\alpha, \beta, \gamma, k, i$ be positive integers such that $\gamma < \alpha \leqslant \beta + \gamma$ and $1 \leqslant i \leqslant p^\beta - 1$. Set

$$\begin{cases} G_{\alpha, \beta, \gamma} = \langle a, b \mid a^{p^\alpha} = b^{p^\beta} = 1,\ b^{-1}ab = a^{1+p^\gamma} \rangle, \\ S_{\alpha, \beta, \gamma, k, i} = \{b^i a, b^i a^e, ..., b^i a^{e^{k-1}}, (b^i a)^{-1}, (b^i a^e)^{-1}, ..., (b^i a^{e^{k-1}})^{-1}\}, \\ \Gamma_{\alpha, \beta, \gamma, k, i} = \mathrm{Cay}(G_{\alpha, \beta, \gamma}, S_{\alpha, \beta, \gamma, k, i}), \end{cases}$$

where e is a positive integer of *order k modulo p^α*; that is, k is the positive integer such that $e^k \equiv 1$ and $e^j \not\equiv 1 \pmod{p^\alpha}$ for all $j < k$.

Clearly, $\Gamma_{\alpha, \beta, \gamma, k, i}$ is a graph of order $p^{\alpha+\beta}$ and valency $2k$, and it will be further shown to be a (p^α, p^β)-metacirculant. The main result of this paper is the following theorem.

THEOREM 1.1. *Let p be a prime and n a positive integer.*

(1) *Let Γ be a connected (p^α, p^β)-metacirculant graph of order p^n and valency less than $2p$. Then Γ is half-transitive if and only if $\Gamma \cong \Gamma_{\alpha, \beta, \gamma, k, i}$ such that $n = \alpha + \beta \geqslant 3$, $k \geqslant 2$ divides $p-1$, and p is coprime to $ie(e-1)$.*

(2) $\Gamma_{\alpha, \beta, \gamma, k, i} \cong \Gamma_{\alpha', \beta', \gamma', k', i'}$ *if and only if* $(\alpha, \beta, \gamma, k) = (\alpha', \beta', \gamma', k')$ *and* $i \equiv i' \pmod{p^{\alpha-\gamma}}$.

(3) *Assume that $n \geqslant 3$. Then for each $k \mid p-1$ with $k > 1$, there exist exactly*

$$(p^{n-2} - 1)/2 + p^{n-3} - 1$$

nonisomorphic connected half-transitive metacirculants of order p^n and valency $2k$.

Xu [32] classified all half-transitive graphs of order p^3 and valency 4. It is known that every vertex-transitive graph of order p^3 is a Cayley graph of a group of order p^3 (see [17]), and clearly every edge-transitive Cayley graph of an abelian group is arc-transitive. By virtue of these facts, we see that the result of [32] is a special case of Theorem 1.1, that is the case where $n = 3$ and $k = 2$. In fact $\Gamma_{2, 1, 1, 2, i}$ are exactly Xu's graphs. On the other hand, by [13, Theorem 1.2], if $n = 3$ then every edge-transitive graph of order p^3 and valency $2p$ is arc-transitive. Thus the conclusion of Theorem 1.1 is the best possible in some sense.

We will give some information about the automorphism groups of our graphs in Section 2 and then prove Theorem 1.1 in Section 3. Throughout the paper, we denote by $G \rtimes H$ a semidirect product of a group G by a group H.

2. AUTOMORPHISM GROUPS

Let Γ be a connected circulant graph of p-power and valency less than $2p$, where p is a prime. In this section, we will prove that Γ is a Cayley graph of a metacyclic group and determine the automorphism group of Γ. First we have a criterion of Sabidussi [24] for a graph to be a Cayley graph, which has by now achieved a folklore status.

LEMMA 2.1 (see [8, Lemma 16.3]). *A graph Γ is a Cayley graph of a group G if and only if G is a subgroup of* $\mathrm{Aut}\,\Gamma$ *and G acts regularly on $V\Gamma$.*

A group G is said to be *metacyclic* if G has a normal cyclic subgroup N such that G/N is cyclic. In particular, the group defined in Definition 2 is metacyclic. The next lemma shows that certain metacirculant graphs are Cayley graphs of metacyclic groups.

LEMMA 2.2. *Let Γ be a metacirculant graph of order p^d, where p is a prime. Assume that $p^{d+1} \nmid |\mathrm{Aut}\,\Gamma|$. Then a Sylow p-subgroup G of* $\mathrm{Aut}\,\Gamma$ *is metacyclic and Γ is a Cayley graph of G.*

Proof. Let σ, τ be as in Definition 1. Then $X := \langle \sigma, \tau \rangle \leqslant \mathrm{Aut}\,\Gamma$ is transitive on $V\Gamma$, and V_i are blocks of imprimitivity of the X-action. Let G be a Sylow p-subgroup of X which contains σ. Then G is transitive on $V\Gamma$. Since $p^{d+1} \nmid |\mathrm{Aut}\,\Gamma|$, it follows that $mn = |V\Gamma| = p^d = |G|$, and so G is regular on $V\Gamma$. By Lemma 2.1, Γ is a Cayley graph of G. Let K be the kernel of X acting on $\{V_1, V_2, ..., V_m\}$. Then $\sigma \in K$, X/K is transitive on $\{V_1, V_2, ..., V_m\}$, and $\langle \tau \rangle K/K \cong \mathbb{Z}_m$ is a Sylow p-subgroup of X/K. Now $G/(G \cap K) \cong GK/K$ is a p-group and transitive on $\{V_1, V_2, ..., V_m\}$. Thus $G/(G \cap K)$ is a Sylow p-subgroup of X/K, and so $G/(G \cap K)$ is a cyclic group of order n. Since $|G| = mn$, $G \cap K$ is of order m. It follows that $G \cap K$ is a Sylow p-subgroup of K. As $\langle \sigma \rangle$ is a Sylow p-subgroup of K, $G \cap K \cong \langle \sigma \rangle$ is cyclic. Therefore, G is a metacyclic group. ∎

For half-transitive metacirculants, we have a consequence:

LEMMA 2.3. *Let Γ be a half-transitive metacirculant graph of order p^d and valency less than $2p$, where p is a prime. Then Γ is a Cayley graph of a metacyclic p-group.*

Proof. Let $A = \mathrm{Aut}\,\Gamma$, and let A_α be the stabilizer in A of $\alpha \in V\Gamma$. Since Γ is half-transitive, A_α has exactly two orbits on $\Gamma(\alpha)$, which are of size less than p. It follows that $p \nmid |A_\alpha|$, and thus $p^{d+1} \nmid |V\Gamma|$. By Lemma 2.2, Γ is a Cayley graph of a metacyclic p-group. ∎

Thus, to prove Theorem 1.1, we only need to consider Cayley graphs of metacyclic p-groups. The next lemma gives information about automorphism groups of certain metacyclic Cayley graphs.

LEMMA 2.4 (see [14, Corollary 1.2]). *Let G be a metacyclic p-group, and let $\Gamma = \mathrm{Cay}(G, S)$ be connected of valency less than $2p$. Then G is a normal subgroup of* $\mathrm{Aut}\,\Gamma$.

The following simple result shows that some of the graph automorphisms of a Cayley graph $\mathrm{Cay}(G, S)$ may be described in terms of group automorphisms of G.

LEMMA 2.5 (see [11, Lemma 2.1]). *Let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley graph, and let* $\mathrm{Aut}(G, S) = \{\sigma \in \mathrm{Aut}(G) \mid S^\sigma = S\}$. *Then* $\mathbf{N}_{\mathrm{Aut}\,\Gamma}(G) = G \rtimes \mathrm{Aut}(G, S)$, *where* $\mathbf{N}_{\mathrm{Aut}\,\Gamma}(G) = \{x \in \mathrm{Aut}\,\Gamma \mid x^{-1}Gx = G\}$.

Combining Lemmas 2.3, 2.4 and 2.5, we have the following result.

LEMMA 2.6. *Let Γ be a half-transitive metacirculant graph of order a p-power and valency less than $2p$. Then $p \geqslant 3$, and there exist a metacyclic p-group G and a subset $S \subset G$ such that*

$$\Gamma \cong \mathrm{Cay}(G, S) \text{ and } \mathrm{Aut}\,\Gamma \cong G \rtimes \mathrm{Aut}(G, S).$$

Thus, to complete the proof of Theorem 1.1, we need some information about the automorphism groups of metacyclic p-groups G. Lindenberg already determined the automorphism group of a metacyclic p-group in [15] and [16]. It follows from a result in [15] that the automorphism group of a finite nonsplit metacyclic p-group is a p-group. Therefore, we only need to consider the remaining case where G is split.

It is easy to show that every finite nonabelian split metacyclic p-group G for odd prime p has a presentation as follows:

$$G = \langle a, b \mid a^{p^\alpha} = 1,\ b^{p^\beta} = 1,\ a^b = a^{1+p^\gamma} \rangle$$

where α, β, γ are positive integers such that $\gamma < \alpha \leqslant \beta + \gamma$. Further, it is easily shown that the parameters α, β, γ uniquely determine the group G up to isomorphism. In the rest of this section, the symbol G always denotes a group with the above presentation. Next we elucidate some information about $\mathrm{Aut}(G)$ which will be used later. First we prove that there exists certain special automorphisms of G.

Let $\mathbf{C}_G(a) = \langle g \in G \mid ga = ag \rangle$, the centralizer of $\langle a \rangle$ in G, and let

$$A(a) := \{ g \in G \mid a^g = a^{1+p^\gamma} \}.$$

Then $\mathbf{C}_G(a) = \langle a, b^{p^{\alpha-\gamma}} \rangle$, and $A(a) = b\mathbf{C}_G(a)$. A straightforward calculation shows that the map

$$a \mapsto a^t, \qquad b \mapsto y,$$

for some integer t, defines an automorphism if and only if $p \nmid t$, $y \in A(a)$ and $y^{p^\beta} = 1$.

Let θ be an automorphism of G such that $\langle a \rangle^\theta = \langle a \rangle$, say $a^\theta = a^t$ for some integer t with $1 \leqslant t < p^\alpha$. Then $b^\theta \in b\mathbf{C}_G(a)$. Since $\mathbf{C}_G(a) = \langle a, b^{p^{\alpha-\gamma}} \rangle$, we have $b^\theta = b^{1+rp^{\alpha-\gamma}} a^s$ for some integers r, s such that $0 \leqslant r < p^{\gamma+\beta-\alpha}$ and $0 \leqslant s < p^\alpha$.

Now let r, s, t be integers such that

$$0 \leqslant r < p^{\beta+\gamma-\alpha}, \qquad 0 \leqslant s < p^\alpha, \qquad 1 \leqslant t < p^\alpha.$$

Then the map

$$a \mapsto a^t, \qquad b \mapsto b^{1+rp^{\alpha-\gamma}} a^s$$

defines an automorphism of G that fixes $\langle a \rangle$ setwise if and only if

$$p \nmid t, \; sp^\beta \equiv 0 \pmod{p^\alpha}.$$

We now summarize the above observation as follows:

LEMMA 2.7. *If θ is an automorphism of G such that $a^\theta \in \langle a \rangle$, then*

$$a^\theta = a^t, \qquad b^\theta = b^{1+rp^{\alpha-\gamma}} a^s$$

for some integers r, s and t such that $0 \leqslant r < p^{\beta+\gamma-\alpha}$, $0 \leqslant s < p^\alpha$, $1 \leqslant t < p^\alpha$, and $p \nmid t$, $sp^\beta \equiv 0 \pmod{p^\alpha}$. Conversely, for any integers r, s and t satisfying the conditions, the map $a \mapsto a^t$, $b \mapsto b^{1+rp^{\alpha-\gamma}} a^s$ defines an automorphism of G that fixes $\langle a \rangle$ setwise.

Next we will determine the order of $\mathrm{Aut}(G)$ and Hall p'-subgroups of $\mathrm{Aut}(G)$. In the following, denote by G' the commutator subgroup of G, namely $\langle y^{-1}x^{-1}yx \mid x, y \in G \rangle$.

THEOREM 2.8. *For an odd prime p, we have that*

$$|\mathrm{Aut}(G)| = (p-1)\, p^{\min(\alpha,\beta)+\min(\beta,\gamma)+\beta+\gamma-1}.$$

Moreover, all Hall p'-subgroups of $\mathrm{Aut}(G)$ *are isomorphic to* $\mathbb{Z}_{p-1}$ *and so are conjugate.*

Proof. Let $N = \{\theta \in \mathrm{Aut}(G) \mid \langle a\rangle^{\theta} = \langle a\rangle\}$, and let

$$\mathscr{K} := \{\langle a\rangle^{\theta} \mid \theta \in \mathrm{Aut}(G)\}.$$

Then $\mathrm{Aut}(G)$ acts transitively on $\mathscr{K}$ and so $|\mathrm{Aut}(G)| = |N|\,|\mathscr{K}|$.

By Lemma 2.7, we have that

$$N = \{\theta_{r,s,t} \mid 0 \leqslant r < p^{\gamma+\beta-\alpha},\ 0 \leqslant s < p^{\alpha},\ 1 \leqslant t \leqslant p^{\alpha},\ p \nmid t,\ sp^{\beta} \equiv 0 \pmod{p^{\alpha}}\},$$

where $\theta_{r,s,t} \in \mathrm{Aut}(G)$ maps a to a^t and b to $b^{1+rp^{\alpha-\gamma}}a^s$. Therefore, we have

$$|N| = (p-1)\, p^{\beta+\gamma-1+\min(\alpha,\beta)}.$$

Let $\mathrm{Hom}(\langle b, G'\rangle/G', \langle a\rangle/G')$ be the set of all homomorphisms from $\langle b, G'\rangle/G'$ to $\langle a\rangle/G'$. We claim that $|\mathscr{K}| = |\mathrm{Hom}(\langle b, G'\rangle/G', \langle a\rangle/G')|$.

It is easily shown that for any $\theta \in \mathrm{Aut}(G)$, $\langle a\rangle^{\theta}$ contains G', and as $\langle b\rangle \cap \langle a\rangle = 1$, we have that $\langle b\rangle \cap \langle a\rangle^{\theta} = 1$. Thus $G = \langle a\rangle^{\theta} \rtimes \langle b\rangle$. It follows that $\langle a\rangle^{\theta}/G'$ is a direct complement of $\langle b, G'\rangle/G'$ in G/G'.

On the other hand, let X/G' be a direct complement of $\langle b, G'\rangle/G'$ in G/G'. Then $\langle b\rangle X = G$ and $\langle b, G'\rangle \cap X = G'$. Let $\Omega_1(G)$ be the subgroup generated by all elements of G of order p. Since $\langle b, G'\rangle$ contains $\Omega_1(G)$ and $\langle b, G'\rangle \cap X = G'$ is cyclic, the subgroup X does not contain $\Omega_1(G)$, and so X is cyclic. Since $G' \leqslant X$, we see that X is also normal in G and $\langle b\rangle \cap X = 1$. Therefore, $G = X \rtimes \langle b\rangle$ is a split metacyclic factorization. It is easily shown that there exist $x \in X$ and $y \in \langle b\rangle$ such that $X = \langle x\rangle$, $\langle b\rangle = \langle y\rangle$, and $y^{-1}xy = x^{1+p^{\gamma}}$. Thus the map $a \mapsto x$, $b \mapsto y$ defines an automorphism of G.

Consequently, there exists a one-to-one correspondence between $\mathscr{K}$ and the set of all complements of $\langle b, G'\rangle/G$ in G/G'. Since G/G' is the direct product of $\langle b, G'\rangle/G'$ and $\langle a\rangle/G'$, it follows from (11.1.2) in [23] that there exists a one-to-one correspondence between $\mathscr{K}$ and $\mathrm{Hom}(\langle b, G'\rangle/G', \langle a\rangle/G')$, so our claim is true.

Observing that $|\langle b, G'\rangle/G'| = p^{\beta}$ and $|\langle a\rangle/G'| = p^{\gamma}$, we have that $|\mathrm{Hom}(\langle b, G'\rangle/G', \langle a\rangle/G')| = p^{\min(\beta,\gamma)}$. Therefore, $|\mathscr{K}| = p^{\min(\beta,\gamma)}$, and so $|\mathrm{Aut}(G)| = (p-1)\, p^{\min(\alpha,\beta)+\min(\beta,\gamma)+\beta+\gamma-1}$.

Finally, since $\langle \theta_{t,0,0} \rangle$ contains a cyclic subgroup of order $p-1$, a Hall p'-subgroup of Aut(G) is isomorphic to $\mathbb{Z}_{p-1}$. Since p is odd, a Sylow 2-subgroup of Aut(G) is cyclic and so Aut(G) is soluble. Therefore, all subgroups of Aut(G) of order $p-1$ are conjugate. ∎

3. PROOF OF THEOREM 1.1

To complete the proof of Theorem 1.1, we first give some properties related to Cayley graphs of $G_{\alpha,\beta,\gamma}$ (see Definition 2). The first lemma is about the connectivity of our graphs.

LEMMA 3.1. *Let $G = G_{\alpha,\beta,\gamma}$ as in Definition 2, and let*

$$R = \{b^i a, b^i a^e, ..., b^i a^{e^{k-1}}\},$$

where $1 \leqslant i \leqslant p^\beta - 1$, $1 < e \leqslant p^\alpha - 1$, and k is the order of e modulo p^α. Then

$$\langle R \rangle = G \text{ if and only if } p \nmid ie(e-1).$$

Proof. Assume that $\langle R \rangle = G$. Suppose that $p \mid e-1$. Let $\Phi(G)$ be the Frattini subgroup of G, and let $\bar{a}$, $\bar{b}$ be the images of a, b in $G/\Phi(G)$, respectively. Then

$$\begin{aligned}\mathbb{Z}_p^2 \cong \langle \bar{a}, \bar{b} \rangle &= G/\Phi(G) = \langle R\Phi(G)/\Phi(G) \rangle = \langle \bar{b}^i \bar{a}, \bar{b}^i \bar{a}^e, ..., \bar{b}^i \bar{a}^{e^{k-1}} \rangle \\ &= \langle \bar{b}^i \bar{a} \rangle \cong \mathbb{Z}_p,\end{aligned}$$

which is a contradiction. Thus p does not divide $e-1$. Similarly, it is easily shown that p divides neither i nor e.

Assume that $p \nmid ie(e-1)$. As $p \nmid e$ and $1 < e \leqslant p^\alpha - 1$, it follows that $k \geqslant 2$, and so $a^{e-1} = (b^i a)^{-1} (b^i a^e) \in \langle R \rangle$. Since $p \nmid e-1$, it follows that $a \in \langle R \rangle$. Thus $b^i = (b^i a)\, a^{-1} \in \langle R \rangle$. Further, as $p \nmid i$, we have that $b \in \langle R \rangle$. Therefore, $G = \langle a, b \rangle = \langle R \rangle$. ∎

The next lemma determines the form of a subset which corresponds to edge-transitive Cayley graphs of the group $G_{\alpha,\beta,\gamma}$.

LEMMA 3.2. *Let $G = G_{\alpha,\beta,\gamma}$, and let T be a generating subset of G of size k. Assume that* Aut(G, T) *is transitive on T and that p does not divide* $|$Aut$(G, T)|$. *Then $k \geqslant 2$, $k \mid (p-1)$, and T is conjugate under* Aut(G) *to*

$$\{b^i a, b^i a^e, ..., b^i a^{e^{k-1}}\},$$

where $1 \leqslant i \leqslant p^\beta - 1$ and $1 < e \leqslant p^\alpha - 1$ such that $p \nmid ie(e-1)$ and the order of e modulo p^α is equal to k.

Proof. Since $\langle T\rangle = G$, $k \geqslant 2$ and so $p \nmid e$. Let θ be the automorphism of G defined by

$$\theta\colon a \mapsto a^{\varepsilon},\ b \mapsto b,$$

where ε is a positive integer of order $p-1$ modulo p^{α}. Then $\langle\theta\rangle \cong \mathbb{Z}_{p-1}$. Since $p \nmid |\mathrm{Aut}(G, T)|$, by Theorem 2.8, $\mathrm{Aut}(G, T)$ is a cyclic subgroup of order dividing $p-1$. As $\langle T\rangle = G$, $\mathrm{Aut}(G, T)$ is faithful on T. Hence $\mathrm{Aut}(G, T)$ is regular on T and $|\mathrm{Aut}(G, T)| = |T| = k$. In particular, $k \mid (p-1)$. Let ρ be such that $\mathrm{Aut}(G, T) = \langle\rho\rangle$. Then

$$T = x^{\langle\rho\rangle} = \{x, x^{\rho}, ..., x^{\rho^{k-1}}\}$$

for some $x \in G$. By Theorem 2.8, there exists $\tau \in \mathrm{Aut}(G)$ such that $\rho^{\tau} = \theta_0 \in \langle\theta\rangle$. Let $y = x^{\tau}$. Noting that $(x^{\rho})^{\tau} = (x^{\tau})^{\rho^{\tau}} = y^{\theta_0}$, it follows that

$$T^{\tau} = \{y, y^{\theta_0}, ..., y^{\theta_0^{k-1}}\}.$$

Now $y = b^i a^j$ for some integers i, j. As $b^{\theta_0} = b$ and $a^{\theta_0} = a^e$ for some e with $1 < e \leqslant p^{\alpha} - 1$, we have that $T^{\tau} = \{b^i a^j, b^i a^{je}, ..., b^i a^{je^{k-1}}\}$. Noting that $\langle T\rangle = G$, a straightforward calculation shows that p divides neither i nor j. Thus there exists an automorphism $\theta_1 \in \mathrm{Aut}(G)$ such that $(a^j)^{\theta_1} = a$ and $b^{\theta_1} = b$, and so

$$T^{\tau\theta_1} = (T^{\tau})^{\theta_1} = \{b^i a, b^i a^e, ..., b^i a^{e^{k-1}}\}.$$

The lemma is proved. ∎

Now we have a result about isomorphisms of Cayley graphs of a p-group.

LEMMA 3.3 [13, Theorem 1.1(3)]. *Let p be a prime, and let G be a p-group. Let* $\mathrm{Cay}(G, S)$ *be a connected Cayley graph of G of valency less than $2p$. Then for any $T \subset G$,* $\mathrm{Cay}(G, T) \cong \mathrm{Cay}(G, S)$ *if and only if $S^{\sigma} = T$ for some σ in* $\mathrm{Aut}(G)$.

Finally we prove Theorem 1.1.

Proof of Theorem 1.1. (1). Let Γ be a connected metacirculant graph of order p^n and of valency less than $2p$.

Assume that $\Gamma \cong \Gamma_{\alpha,\beta,\gamma,k,i}$ such that $n = \alpha + \beta \geqslant 3$ and $p \nmid ie(e-1)$, where $1 < e \leqslant p^{\alpha} - 1$ and $1 \leqslant i \leqslant (p^{\beta} - 1)/2$. Let $R = \{b^i a, b^i a^e, ..., b^i a^{e^{k-1}}\}$, and write $G = G_{\alpha,\beta,\gamma}$ and $S = S_{\alpha,\beta,\gamma,k,i}$. Then $S = R \cup R^{-1}$ and $\Gamma \cong \mathrm{Cay}(G, S)$. It is easily shown that there exists $\rho \in \mathrm{Aut}(G)$ such that $\rho\colon a \mapsto a^e$, $b \mapsto b$. Now $\langle\rho\rangle \leqslant \mathrm{Aut}(G, R)$ and $\langle\rho\rangle$ is transitive on R. It follows that Γ is edge-transitive. Suppose that Γ is arc-transitive. By Lemma 2.6,

$\operatorname{Aut}\Gamma = G \rtimes \operatorname{Aut}(G, S)$, and thus $\operatorname{Aut}(G, S)$ is transitive on S. Since Γ is connected, $\langle S \rangle = G$, and hence by Lemma 3.2, S is conjugate to a subset of the form $\{b^{i_0}a, b^{i_0}a^{e_0}, ..., b^{i_0}a^{e_0^{k_0-1}}\}$. However, $(b^i a)^{-1} = a^{-1}b^{-i} = b^{-i}.\, b^i a^{-1} b^{-i} = b^{-i}a^l$ for some integer l. Since $1 \leqslant i \leqslant (p^\beta - 1)/2$, $b^{-i}a^l \neq b^i a^h$ for any integer h, which is a contradiction. Therefore, Γ is half-transitive.

Assume now that Γ is half-transitive. Then by Lemma 2.3, $\Gamma \cong \operatorname{Cay}(G_{\alpha,\beta,\gamma}, T)$ for some positive integers α, β, γ such that $n = \alpha + \beta \geqslant 3$ and for some $T \subset G_{\alpha,\beta,\gamma}$ such that $\langle T \rangle = G_{\alpha,\beta,\gamma}$, $T = T^{-1}$ and $|T| < 2p$. Let $G = G_{\alpha,\beta,\gamma}$. By Lemma 2.6, $\operatorname{Aut}\Gamma = G \rtimes \operatorname{Aut}(G, T)$. Since Γ is half-transitive, $\operatorname{Aut}(G, T)$ on T has exactly two orbits T_1 and T_2 of the same size in its action on T. Thus $\operatorname{Aut}(G, T_1)$ is transitive on T_1. By Lemma 3.2, T_1 is conjugate in $\operatorname{Aut}(G)$ to $\{b^i a, b^i a^e, ..., b^i a^{e^{k-1}}\}$ for some suitable integers e, i, k, that is, there exists $\rho \in \operatorname{Aut}(G)$ such that $T_1^\rho = \{b^i a, b^i a^e, ..., b^i a^{e^{k-1}}\}$. Since $T = T^{-1}$, it follows that $T^\rho = (T^\rho)^{-1}$, and thus $(b^i a^{e^j})^{-1} \in T^\rho$ for $0 \leqslant j \leqslant k-1$. Further, it is easily shown that $(b^i a^{e^j})^{-1} \notin T_1^\rho$. It then follows that $(T_1^\rho)^{-1} = T_2^\rho$. Thus $T^\rho = S_{\alpha,\beta,\gamma,k,i}$, and so $\Gamma \cong \Gamma_{\alpha,\beta,\gamma,k,i}$, as in part (1).

(2) By Lemma 2.2, $G_{\alpha,\beta,\gamma}$ is a Sylow p-subgroup of the automorphism group of the corresponding Cayley graph $\Gamma_{\alpha,\beta,\gamma,k,i}$. It follows that $\Gamma_{\alpha,\beta,\gamma,k,i} \cong \Gamma_{\alpha',\beta',\gamma',k',i}$ if and only if $(\alpha, \beta, \gamma, k) = (\alpha', \beta', \gamma', k')$.

For convenience, write $G = G_{\alpha,\beta,\gamma}$ where $1 \leqslant \gamma < \alpha \leqslant \beta + \gamma$. Let $S_i = S_{\alpha,\beta,\gamma,k,i}$ and $\Gamma_i = \operatorname{Cay}(G, S_i)$ where $1 \leqslant i \leqslant (p^\beta - 1)/2$. We claim that for any $i, j \in \{1, 2, ..., (p^\beta - 1)/2\}$,

$$\Gamma_i \cong \Gamma_j \text{ if and only if } i \equiv j \pmod{p^{\alpha-\gamma}}.$$

Assume first that $i \equiv j \pmod{p^{\alpha-\gamma}}$, say $j = hp^{\alpha-\gamma} + i$ for some integer h. Let i' be an integer such that $i'i \equiv 1 \pmod{p^{\beta-(\alpha-\gamma)}}$. Then it is easily shown that $b^{-hi'p^{\alpha-\gamma}} a b^{hi'p^{\alpha-\gamma}} = a$. It follows that there exists $\sigma \in \operatorname{Aut}(G)$ such that $a^\sigma = a$ and $b^\sigma = b^{1+hi'p^{\alpha-\gamma}}$. Now for any integer l, we have that $(b^i a^l)^\sigma = b^{i(1+hi'p^{\alpha-\gamma})} a^l = b^{i+hii'p^{\alpha-\gamma}} a^l = b^{i+hp^{\alpha-\gamma}} a^l = b^j a^l$. It follows that $S_i^\sigma = S_j$, and $\Gamma_i \cong \Gamma_j$.

Assume now that $\Gamma_i \cong \Gamma_j$. Then by Lemma 3.3, there exists $\sigma \in \operatorname{Aut}(G)$ such that $S_i^\sigma = S_j$. Let $R_i = \{b^i a, b^i a^e, ..., b^i a^{e^{k-1}}\}$. As Γ_i and Γ_j are half-transitive, it follows from Lemma 2.6 that $\operatorname{Aut}(G, S_i)$ has exactly two orbits R_i and R_i^{-1}, and that $\operatorname{Aut}(G, S_j)$ has exactly two orbits R_j and R_j^{-1}. Thus $R_i^\sigma = R_j$ or R_j^{-1}. Suppose first that $R_i^\sigma = R_j$. Then $(b^i a)^\sigma = b^j a^{e_1}$ and $(b^i a^e)^\sigma = b^j a^{e_2}$ for some integers e_1, e_2. Thus σ maps $a^{e-1} = (b^i a)^{-1} b^i a^e$ to $(b^j a^{e_1})^{-1} b^j a^{e_2} = a^{e_2 - e_1}$. Since $p \nmid e-1$, $a^\sigma = a^t$ for some integer t, and thus

$$\sigma\colon b^i = b^i a a^{-1} \mapsto b^j a a^{-t} = b^j a^{1-t}.$$

On the other hand, since $\langle a\rangle^{\sigma}=\langle a\rangle$, it follows from Lemma 2.7 that $b\to b^{1+rp^{\alpha-\gamma}}a^{s}$ for some integers r, s. Thus

$$b^{i}\mapsto(b^{1+rp^{\alpha-\gamma}}a^{s})^{i}=b^{i+irp^{\alpha-\gamma}}a^{v},$$

where v is an integer. It follows that $b^{i+irp^{\alpha-\gamma}}=b^{j}$. Thus $i+irp^{\alpha-\gamma}\equiv j$ (mod p^{β}). Since $\alpha-\gamma\leqslant\beta$, we have that $i\equiv j$ (mod $p^{\alpha-\gamma}$). Suppose now that $R_i^{\sigma}=R_j^{-1}$. Then $(b^{i}a)^{\sigma}=b^{-j}a^{u}$ and $(b^{i}a^{e})^{\sigma}=b^{-j}a^{v}$ for some integers u, v. A similar argument as in the previous case leads to $i\equiv j$ (mod $p^{\alpha-\gamma}$), as claimed.

(3) By part (2), for a given group $G_{\alpha,\beta,\gamma}$ with parameters α,β,γ, noting that $(R_i)^{-1}=R_{p^{\beta}-i}$, the number $\mathbf{n}$ of nonisomorphic graphs $\Gamma_{\alpha,\beta,\gamma,k,i}$ of valency $2k$ is equal to

$$|\{i\mid p\nmid i,\ 1\leqslant i\leqslant\min\{(p^{\beta}-1)/2,\ p^{\alpha-\gamma}-1\}\}|.$$

If $\alpha-\gamma<\beta$ then $p^{\alpha-\gamma}-1<(p^{\beta}-1)/2$. It follows that

$$\mathbf{n}=\begin{cases}p^{\beta-1}(p-1)/2, & \text{if } \alpha-\gamma=\beta;\\ p^{\alpha-\gamma-1}(p-1), & \text{if } \alpha-\gamma<\beta.\end{cases}$$

Since $0<\gamma<\alpha\leqslant\beta+\gamma$ and $\alpha+\beta=n$, we have that $\beta-1$ may run through $\{0,...,n-3\}$ and that if $\alpha-\gamma<\beta$ then $\alpha-\gamma-1$ may run through $\{0,...,n-4\}$. Recall that the group $G_{\alpha,\beta,\gamma}$ is uniquely determined by the parameters α,β,γ. By part (2), if $(\alpha,\beta,\gamma)\neq(\alpha',\beta',\gamma')$, then $\Gamma_{\alpha,\beta,\gamma,k,i}\ncong\Gamma_{\alpha',\beta',\gamma',k',i'}$. Consequently, the number of nonisomorphic metacirculants of order p^{n} and valency $2k$ is equal to

$$\sum_{0\leqslant\beta-1\leqslant n-3}p^{\beta-1}(p-1)/2+\sum_{0\leqslant\alpha-\gamma-1\leqslant n-4}p^{\alpha-\gamma-1}(p-1)$$
$$=(p^{n-2}-1)/2+p^{n-3}-1.$$

This completes the proof of the theorem. ▮

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